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Pseudoconvexity and general relativity

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1. INTRODUCTION

Global hayperbolicity is well known to play an important in Lorentzian geometry and General Relativity. Seifert [16] proved that it was a sufficient condition for the existence of maximal length geodesic segments joining causally related points. This gave a partial generalization of the important Hopf-Rinow Theorem (cf. O'Neill [15]) to Lorentzian manifolds. Georch [9] proved that a space has a global Cauchy surface S iff it is a globally hyperbolic spacetime. Furthermore, Geroch showed that these spacetimes are topological products of the form $\mathbb{R} \times S$ and that global hyperbolicity is a stable property. An important application of global hyperbolicity is given in the singularity theorems. Hawking and Penrose [11] showed that many spacetimes have large globally hyperbolic subsets, and they used these subsets to construct causal geodesics without conjugate points. The timelike convergence condition and the generic condition were then used to deduce the incompleteness of these geodesics. The existence of an incomplete causal geodesic is usually taken as indicating a physical singularity.

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In this paper we consider a generalization of global hyperbolicity called pseudoconvexity. The primary motivations for this paper are to explain how pseudoconvexity can often be used in place of global hyperbolicity and to explain some of the implications for geodesic structure this assumption entails.

A spacetime (M, g) is said to be causally pseudoconvex iff given any compact set K in M there is always a larger compact set K' such that all causal geodesic segments joining points of K lie entirely in K'. This basic concept can be used for any class of geodesics. For example, null pseudoconvexity is the requirement that all null geodesic segments with endpoints in K lie entirely in K'.

Like global hyperbolicity, (causal) pseudoconvexity is a type of completeness requirement. Intuitively, one may think of pseudoconvex spaces as failing to have any «interior» points missing. Thus, Minkowski space less any compact set is neither globally hyperbolic nor causally pseudoconvex. A simple example of a causally pseudoconvex spacetime which is not globally hyperbolic is the open strip $a < x < b$ in the Minkowski (t, x) plane. We say that causal pseudoconvexity generalizes global hyperbolicity since every globally hyperbolic spacetime is causally pseudoconvex.

In the theory of pseudodifferential equations, the concept of pseudoconvexity is applied to the bicharacteristic segments in the study of global solvability. If (M, g) is a Lorentzian manifold with d'Alembertian $\Box$, then the symbol of $\Box$ is the metric tensor in the contravariant form (cf. Trèves [17]). In this case the bicharacteristic segments are the null geodesic segments, and the inhomogeneous wave equation

$$\Box u = f$$

has global solutions in the distribution sense if (1) (M, g) is null pseudoconvex and (2) each end of each inextendible null geodesic fails to be imprisioned. We call this second requirement disprisonment of null geodesics. In the language of PDE's, it is the requirement that the operator be of real principal type.

Interestingly, both null pseudoconvexity and disprisonment of null geodesics fail to be separately C^1-stable in the Whitney topology, but the requirement that they hold jointly *is* C^1-stable (cf. [5, p. 18]).

As one would expect, disprisonment and pseudoconvexity have important implications for the geodesic structure of a spacetime. Williams [18] found examples of geodesically complete spacetimes with arbitrarily close incomplete metrics in the Whitney C^r-topology. Hence geodesic completeness fails to be C^r-stable for all $r \geqslant 1$. On the other hand, Beem and Ehrlich [2] have established the C^1-fine stability of causal geodesic completeness for Lorentzian manifolds which are both causally pseudoconvex and causally disprisoning. As a corollary, if (M, g) is a causally geodesically complete and globally hyperbolic spacetime, then there is a C^1-fine neighborhood $U(g_0)$ of g_0 in the space of all Lorentzian metrics on M such that each $g \in U(g_0)$ is causally geodesically complete.

Let $\{\gamma_n\}$ be a sequence of inextendible geodesics and assume that $\gamma_n'(0) \to \gamma'(0)$ where γ is also an inextendible geodesic. Let lim inf and lim sup represent the lower and upper Hausdorff limits, cf. Section II and [1, p. 34]. In general, one has that the image of γ lies in the lim inf of the images of the γ_n and this in turn lies in the lim sup of the images of the γ_n:

$$\gamma \subseteq \liminf \gamma_n \subseteq \limsup \gamma_n .$$

For arbitrary spacetimes, these three sets may be distinct. On the other hand, we show that if (M, g) is pseudoconvex and disprisoning, then these sets are all equal.

2. PRELIMINARIES

Let (M, g) be an n-dimensional spacetime of signature $(+, -, -, \ldots, -)$. For each tangent vector v, there is a unique inextendible geodesic $\gamma : (a, b) \to \to (M, g)$ with $\gamma'(0) = v$. If this geodesic is denoted by γ_v, then the exponential map

$$\exp_p : T_pM \to M$$

is defined by $\exp_p(v) = \gamma_v(1)$ provided $\gamma_v(1)$ exists. If every geodesic starting at p is complete, then the domain of $\exp_p$ is all of T_pM. Otherwise, $\exp_p$ is only defined on a proper subset of T_pM.

Conjugate points are defined using the derivative of the exponential map. If $v \in T_pM$, then $\exp_{p*} : T_v(T_pM) \to T_qM$ where $q = \exp_p(v)$. If $\exp_{p*}$ has rank $< n$ at v, then the point q is said be conjugate to p along the geodesic $\gamma_v(t) = = \exp_p(tv)$.

Let $\mathrm{Lor}(M)$ denote the space of all Lorentzian metrics on M (cf. [1], [5], [13]). We define the Whitney C^r topologies on $\mathrm{Lor}(M)$ using a fixed countable covering B_i of M by compact sets which lie in coordinate neighborhoods and form a locally finite cover. Assume $\delta : M \to (0, +\infty)$ is a continuous function. Let $g_1, g_2 \in \mathrm{Lor}(M)$; then $|g_1 - g_2|_r < \delta$ if, for each $p \in M$, all of the corresponding coefficients and their mixed derivatives up to order r of the two metric tensors g_1 and g_2 are $\delta(p)$-close at p when calculated in the coordinates of all B_i which contain p. The sets $U(g_1, \delta) = \{g \in \mathrm{Lor}(M) | \cdot | g_1 - g|_r < \delta\}$ for arbitrary $g_1 \in \mathrm{Lor}(M)$ and continuous $\delta : M \to (0, +\infty)$ form a basis for the C^r--fine topology on $\mathrm{Lor}(M)$. This topology is independent of the choice of $\{B_i\}$.

In order to define the Hausdorff upper and lower limits, let A_n be an arbitrary sequence of subsets of M. Then

$$\limsup\{A_n\} = \{p \in M \mid \text{every neighborhood of } p \text{ intersects infinitely many } A_n\}$$

and

$$\liminf\{A_n\} = \{p \in M \mid \text{every neighborhood of } p \text{ intersects all but a finite number of the } A_n\}.$$

The Hausdorff upper and lower limits always exist and are always closed subsets of M; however, one or both of them may be empty. Clearly, $\liminf\{A_n\} \subseteq \subseteq \limsup\{A_n\}$.

We now define causal (geodesic) pseudoconvexity. Similar definitions hold for null (geodesic) pseudoconvexity, etc.

DEFINITION 2.1. *The spacetime* (M, g) *is* causally pseudoconvex *iff for each compact set* K *in* M *there is a compact set* K' *such that each casual geodesic segment* $\gamma : [a, b] \to M$ *with* $\gamma(a), \gamma(b) \in K$ *satisfies* $\gamma\,[a, b] \subseteq K'$.

We now define causal geodesic disprisonment. Similar definitions hold for null geodesic disprisonment etc.

DEFINITION 2.2. *The spacetime* (M, g) *has* no imprisoned causal geodesics *or is* causally disprisoning *iff for each inextendible causal geodesic* $\gamma : (a, b) \to M$ *and* $t_0 \in (a, b)$, *both of the sets* $\{\gamma(t) \mid a < t \leqslant t_0\}$ *and* $\{\gamma(t) \mid t_0 \leqslant t < b\}$ *fail to have compact closure.*

The definition is the requirement that neither end of γ be (totally) imprisoned in a compact set. When (M, g) satisfies both the causal pseudoconvexity and causal disprisoning conditions, then it is easy to show that if K is compact and $\gamma : (a, b) \to M$ is an inextendible causal geodesic, then there are parameter values t_1 and t_2 in (a, b) with $a < t_1 < t_2 < b$ such that $\gamma(t)$ is not in K for all $a < t < t_1$ and $t_2 < t < b$. If (M, g) is strongly causal, then (M, g) is causally disprisoning but not necessarily pseudoconvex. Minkowski spacetime with a point removed is causally disprisoning but not causally pseudoconvex.

Most of our notational conventions are standard and may be found in [1], [10], and [15].

3. GEODESICS AND LIMITS

In this section we consider the limit sets of sequences of geodesics. Let γ_n be a sequence of inextendible geodesics and assume $\gamma_n'(0) \to \gamma'(0)$ where γ is also an inextendible geodesic. For each t_0 in the domain of γ, we have t_0 in the domain of γ_n for sufficiently large n and $\gamma_n(t_0) \to \gamma(t_0)$. Thus, letting γ and γ_n denote the image sets as well as the mappings, we have $\gamma \subseteq \lim\inf \{\gamma_n\}$. In general, this containment is proper. For example, let $M = \{(x, t) \mid -\pi/2 < x < \pi/2\}$ be two dimensional universal anti-de Sitter spacetime with metric $ds^2 = \sec^2(x) \cdot \cdot (dt^2 - dx^2)$ and time orientation determined by $\frac{\partial}{\partial t}$; cf. [1, p. 141]. If $v_p = \gamma'(0)$ is a null vector at some $p \in M$ and γ_n are timelike geodesics with $\gamma_n'(0) \to v_p$, then the image of γ will be a null geodesic and thus a segment at 45° lying in the strip $-\pi/2 < x < \pi/2$. On the other hand, it is easy to check that the images of the γ_n approach an entire sequence of null geodesics. The lim inf of $\{\gamma_n\}$ is equal to a sequence of straight line semgents lying in the strip $-\pi/2 < x < \pi/2$. These line segments alternate in Euclidean slope between + 1 and − 1 and have endpoints on the «edges» $x = +\pi/2$ and $x = -\pi/2$. Thus $\gamma \neq \lim\inf \{\gamma_n\}$. Since this example is conformal to the open strip $-\pi/2 < x < \pi/2$ in the Minkowski plane, this type of behavior occurs in causally simple (hence also causally continuous and strongly causal) spacetimes. In the given example one has $\lim\inf \{\gamma_n\} = \lim\sup \{\gamma_n\}$, but it is easy to construct examples with proper inclusions $\gamma \subset \lim\inf \{\gamma_n\} \subset \lim\sup \{\gamma_n\}$.

On the other hand, one can establish equality for sequences $\{\gamma_n\}$ of causal geodesics in spaces which are both causally pseudoconvex and causally disprisoning.

PROPOSITION 3.1. *Let (M, g) be a spacetime which is both causally pseudoconvex and causally disprisoning. If $\{\gamma_n\}$ are inextendible causal geodesics with $\gamma_n'(0) \to \gamma'(0)$ where γ is also an inextendible (causal) geodesic, then $\gamma = \lim\inf \{\gamma_n\} = \lim\sup \{\gamma_n\}$.*

Proof. It is sufficient to show that $\lim\sup \{\gamma_n\} \subseteq \gamma$. To this end, assume that $q \in \lim\sup \{\gamma_n\}$. Then there is some subsequence $\{\gamma_m\}$ of $\{\gamma_n\}$ and a corresponding sequence $\{q_m\}$ with $q_m \in \gamma_m$ for each m and $q_m \to q$. Assume without loss of generality that $q_m = \gamma_m(t_m)$ for $t_m > 0$. Choose any compact set K containing all of $\gamma(0)$, $\gamma_m(0)$, q and q_m. By causal pseudoconvexity, there is a compact set K' containing all of the causal geodesic segments $\gamma_m \mid [0, t_m]$. The limit geodesic γ must be causal because $\gamma_m'(0) \to \gamma'(0)$. Using causal disprisonment, one obtains a value $T > 0$ such that $\gamma(T) \in M - K'$. Then $\gamma_m(T) \to \gamma(T)$ and $\gamma_m \mid [0, t_m] \subseteq K'$ yield $0 < t_m < T$ for all sufficiently large m. Using the compactness of the interval $[0, T]$, one obtains a subsequence $\{\gamma_k\}$ of $\{\gamma_m\}$ with the corresponding subsequence $t_k \to \tau \in (a, T)$. Then $q_k = \gamma_k(t_k) \to \gamma(\tau)$ shows that $q = \gamma(\tau)$, and thus that $\lim\sup \{\gamma_n\} \subseteq \gamma$ as desired. ∎

We now show that global hyperbolicity implies causal pseudoconvexity. Recall that a strongly causal spacetime is globally hyperbolic iff $J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$, cf. [10].

LEMMA 3.2. *If (M, g) is a globally hyperbolic spactime, then (M, g) is both causally pseudoconvex and causally disprisoning.*

Proof. Causal disprisoning follows from the fact that in a strongly causal spacetime, any inextendible causal curve must eventually leave and never return to a compact set. To show causal pseudoconvexity, let K be a given compact set. We may construct a finite number of sets of the form $J^+(p_i) \cap J^-(q_i)$, say for $1 \leqslant i \leqslant k$, such that K lies in the union of the interiors of these sets. Set

$$K' = \bigcup_{j=1}^{k} \bigcup_{i=1}^{k} [J^+(p_i) \cap J^-(q_j)].$$

Let $c : [a, b] \to M$ be any future directed causal curve with $c(a), c(b) \in K$. Choose fixed i and j such that $c(a) \in J^+(p_i)$ and $c(b) \in J^-(q_j)$. Then $q_j \in J^+(p_i)$ and the entire curve $c[a, b]$ must lie in $J^+(p_i) \cap J^-(q_j)$, hence also in K'. Since this holds for arbitrary future directed causal curves, it holds for all causal geodesic segments with endpoints in K'. ∎

COROLLARY 3.3. *Let (M, g) be a globally hyperbolic spacetime. If $\{\gamma_n\}$ are inextendible causal geodesics with $\gamma_n'(0) \to \gamma'(0)$ where γ is an inextendible geodesic, then $\gamma = \lim\inf \{\gamma_n\} = \lim\sup \{\gamma_n\}$.* ∎

It is important to realize in Proposition 3.1 and Corollary 3.3 that the sequence γ_n must be causal. Consider, for example, two dimensional universal anti-de Sitter space with space and time interchanged. Thus $M = \{(t, x) \mid -\pi/2 < x < \pi/2\}$ with $ds^2 = \sec^2(x)(-dt^2 + dx^2)$ and time orientation determined by the timelike vector field $\frac{\partial}{\partial x}$. This spacetime is globally hyperbolic. The sequence of geodesics $\{\gamma_n\}$ considered at the beginning of this section in anti-de Sitter space are now spacelike and show that the conclusion of Theorem 3.3 may fail for spacelike sequences in globally hyperbolic spacetimes.

One can, however, extend Proposition 3.1 to spacelike geodesics by requiring that the pseudoconvexity and disprisoning assumptions hold for spacelike as well as causal geodesics. We say (M, g) is geodesically pseudoconvex iff for each compact K, all geodesic segments with endpoints in K lie in some compact K' as before. It is important to note that K' depends only on K.

THEOREM 3.4. *Let (M, g) be a spacetime which is geodesically pseudoconvex and which has no imprisoned geodesics of any type. If $\{\gamma_n\}$ is a sequence of inextendible geodesics with $\gamma_n'(0) \to \gamma'(0)$ where γ is an inextendible geodesic, then $\gamma = \liminf\{\gamma_n\} = \limsup\{\gamma_n\}$.* ■

4. PSEUDOCONVEXITY

In this section we discuss some of our results involving pseudoconvexity which are of interest in Lorentzian geometry and General Relativity. We begin by stating a stability result (cf. Proposition 4.5 of [5]).

PROPOSITION 4.1. *Let (M, g) be a spacetime which is both causally pseudoconvex and causally disprisoning. Then there is a C^1-fine neighborhood $U(g)$ of g in the space of Lorentzian metrics Lor (M) such that each $g' \in U(g)$ is both causally pseudoconvex and causally disprisoning.* ■

This result is the pseudoconvex analogue of a result of Geroch [9, p. 448] which guarantees the C^0-stability of global hyperbolicity. Since global hyperbolicity yields the existence of solutions to the Cauchy problem in General Relativity, the Geroch result implies a stability of solvability. Studies of the Cauchy problem in General Relativity may be found in Chapter 7 of [10] and in [14]. The concept of pseudoconvexity first arose in modern studies of global solvability of (partial) differential equations, and a brief digression may be useful here.

The wave equation on a spacetime is a primary example of a hyperbolic equation. Recall that two problems with the hyperbolic equation $Pu = f$ are: (1) it may be solvable (even with distributions allowed for u) only for some f; (2) unlike elliptic equations, u need not be as smooth as f (e.g., $f \in C^\infty$ need not imply $u \in C^\infty$). This forces the use of distributions and thus the need for regularity (smoothness) theorems for solutions. Thus one analyzes solvability in three steps: (1) establish the existence of a parametrix (i.e., an E such that PE is the identity mod C^∞); (2) obtain an exact solution; and (3) study the regularity of the solution. Given the existence of a parametrix, classical local constructions due to Hadamard handle (2) and (3) for the wave equation (see e.g. [8]), and analogous techniques exist for general hyperbolic equations.

We note in passing that C^r regularity is *much* more difficult to study, usually requiring a great deal of abstract P.D.E. theory including Sobolev spaces and

Fourier integral operators. It is also interesting to note that classical «hard» analysis appears mostly in (2) and (3), while the most difficult step (1) is predominantly «soft» analysis.

Accordingly, one studies global solvability by investigating (1): the global existence of a parametrix. There have been two main results. Leray [12] showed that global hyperbolicity sufficed, and Duistermaat and Hörmander [7] showed that null pseudoconvexity and null disprisonment (jointly) sufficed. These last two conditions represent a genuine improvement since global hyperbolicity implies them but the converse is not true. (Examples exist [7], however, to show that these conditions are not even individually necessary; more work is still needed there). Geroch [9] provided the topological meaning of global hyperbolicity, and some geometrical meaning has been obtained via conformal invariance [1, p. 107]. The goal in our series of papers [3], [4], [5], [6] is to elucidate the geometrical meaning of pseudoconvexity and disprisonment for various types of geodesic systems and geometries.

Returning to the main discussion, we now give the corresponding stability result for solvability in the pseudoconvex case. Here solvability is in the distribution sense (cf. Corollary 3.4 of [5]).

COROLLARY 4.2. *Let (M, g) be a spacetime which is both causally pseudoconvex and causally disprisoning. Then there is a C^1-fine neighborhood $U(g)$ of g in $Lor(M)$ such that for each $g' \in U(g)$ the equation $\Box' u \in f + C^\infty(M)$ has global (distribution) solutions u. Here $\Box'$ is the d'Alembertian for g' and f is any distribution such that $\Box u \in f + C^\infty(M)$ is solvable.* ∎

It might be of some interest to have a careful analysis of the corresponding stability of exact solvability and regularity (of Hadamard's methods, for example), although their local nature tends to lead one to believe that no additional difficulties would be encountered.

Seifert [16] has shown that for globally hyperbolic spacetimes, any two causally related points may be joined by a causal geodesic. His argument is to let $q \in J^+(p)$ and consider the space $C(p, q)$ of future directed timelike curves from p to q. This space $C(p, q)$ is compact using the C^0 topology on curves and the length functional $L : C(p, q) \to \mathbb{R}$ is upper semicontinuous for this topology. It follows that L attains a maximum on some $\gamma_0 \in C(p, q)$ and this γ_0 is the desired maximal length timelike geodesic from p to q.

Seifert's theorem is the strongest possible within the class of globally hyperbolic spacetimes. In general, there is no geodesic joining a pair of points in a globally hyperbolic spacetime unless these points are causally related. For example, returning to universal anti-de Sitter space with space and time interchanged, we obtain a globally hyperbolic spacetime with many spacelike related point pairs not joined by any geodesic.

A pseusoconvex extension of Seifert's Theorem has recently been obtained [6]. This result, a pseudoriemannian version of the Hopf-Rinow Theorem, yields the existence of at least one geodesic joining any given pair of points for a certain class of spacetimes which are pseudoconvex and disprisoning for *all* types of geodesics.

THEOREM 4.3. *Let (M, g) be a space-time which is geodesically pseudoconvex and which has no imprisoned geodesics of any type. If (M, g) has no conjugate points, then given any pair of points there is at least one geodesic joining them.* ∎

As a last application of pseudoconvexity, we state some recently obtained results [2] on the stability of geodesic completeness. These results were motivated by the example of Williams [18] showing geodesic completeness is *not* stable for spacetimes in general. Previously, it had been throught that geodesic completeness was stable [13], [1, p. 175].

THEOREM 4.4. *Let (M, g) be a spacetime which is causally pseudoconvex and causally disprisoning. If (M, g) is causally geodesically complete, then there is a C^1-fine neigborhood $U(g)$ of g in Lor (M) such that each $g' \in U(g)$ is also causally geodesically complete.* ∎

COROLLARY 4.5. *If (M, g) is a globally hyperbolic spacetime which is causally complete, then there is a C^1-fine neighborhood $U(g)$ of g such that all $g' \in U(g)$ are causally geodesically complete.* ∎

For Minkowski spacetime the above corollary can be strengthened to include spacelike completeness.

THEOREM 4.6. *Let $(\mathbb{R}^n, \eta)$ be Minkowski spacetime. There exists a C^1-fine neighborhood $U(\eta)$ of η in Lor $(\mathbb{R}^n)$ such that* all *geodesics of* each $g' \in U(g)$ *are complete.* ∎

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